Differentiation is restricted to directions tangent to the surface. Consequently the vector $\nabla(\mathbf{M}\cdot\mathbf{n})$ lies in the surface, while the vector $\nabla\times(\mathbf{n}\times\mathbf{M})$ is normal to it. *The discontinuities exhibited by the vector* **B** *in its transition through a surface distribution of magnetic moment can be expressed by the single formula:*

$$\text{(29)}\qquad \mathbf{B}_+ - \mathbf{B}_- = -\mu_0[\nabla(\mathbf{n}\cdot\mathbf{M}) + \nabla\times(\mathbf{n}\times\mathbf{M})].$$

INTEGRATION OF THE EQUATION $\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J}$

4.14. Vector Analogue of Green's Theorem.—The classical treatment of the vector potential is based on a resolution into rectangular components. On the assumption that $\nabla\cdot\mathbf{A} = 0$, each component can be shown to satisfy Poisson's equation and the methods developed for the analysis of the electrostatic potential are applicable.

The possibility of integrating the equation $\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J}$ directly by means of a set of vector identities wholly analogous to those of Green for scalar functions appears to have been overlooked. Let V be a closed region of space bounded by a regular surface S, and let **P** and **Q** be two vector functions of position which together with their first and second derivatives are continuous throughout V and on the surface S. Then, if the divergence theorem be applied to the vector $\mathbf{P}\times\nabla\times\mathbf{Q}$, we have

$$\text{(1)}\qquad \int_V \nabla\cdot(\mathbf{P}\times\nabla\times\mathbf{Q})\,dv = \int_S (\mathbf{P}\times\nabla\times\mathbf{Q})\cdot\mathbf{n}\,da.$$

Upon expanding the integrand of the volume integral one obtains the vector analogue of Green's first identity, page 165,

$$\text{(2)}\qquad \int_V (\nabla\times\mathbf{P}\cdot\nabla\times\mathbf{Q} - \mathbf{P}\cdot\nabla\times\nabla\times\mathbf{Q})\,dv = \int_S (\mathbf{P}\times\nabla\times\mathbf{Q})\cdot\mathbf{n}\,da.$$

The analogue of Green's second identity is obtained by an interchange of the roles of **P** and **Q** in (2) followed by subtraction from (2). As a result

$$\text{(3)}\qquad \int_V (\mathbf{Q}\cdot\nabla\times\nabla\times\mathbf{P} - \mathbf{P}\cdot\nabla\times\nabla\times\mathbf{Q})\,dv = \int_S (\mathbf{P}\times\nabla\times\mathbf{Q} - \mathbf{Q}\times\nabla\times\mathbf{P})\cdot\mathbf{n}\,da.$$

4.15. Application to the Vector Potential.—We shall assume that the volume density of current $\mathbf{J}(x, y, z)$ is a bounded but otherwise arbitrary function of position. The regular surface S bounding a volume V need not necessarily contain within it the entire source distribution, or even any part of it. As in Sec. 3.4 we shall choose O as an arbitrary origin and $x = x'$, $y = y'$, $z = z'$ as a fixed point within V.

Now let **P** represent the vector potential **A** subject to the conditions

$$\text{(4)}\qquad \nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J}, \qquad \nabla\cdot\mathbf{A} = 0,$$

where μ is the permeability of a medium assumed homogeneous and isotropic. The choice of Q may be made in either of two ways. It will be recalled that in the scalar case the Green's function ψ satisfied Laplace's equation $\nabla^2\psi = 0$ and could be interpreted as the potential at (x', y', z') due to a charge $4\pi\epsilon$ located at (x, y, z). Likewise the vectorial Green's function **Q** may be chosen to represent the vector potential at (x', y', z') produced by a current of density $4\pi/\mu$ located at (x, y, z) and directed arbitrarily along a line determined by the unit vector **a**.

$$\text{(5)}\qquad \mathbf{Q}(x, y, z; x', y', z') = \frac{\mathbf{a}}{r}, \qquad r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}.$$

However the divergence of this function is not zero and consequently (5) fails to satisfy the condition $\nabla\times\nabla\times\mathbf{Q} = 0$. On the other hand, the vector potential arising from a distribution of magnetic moment has been shown to satisfy $\nabla\times\nabla\times\mathbf{A} = 0$ everywhere and an appropriate Green's function for the present problem is, therefore,

$$\text{(6)}\qquad \mathbf{Q} = \nabla\left(\frac{1}{r}\right)\times\mathbf{a} = \nabla\times\frac{\mathbf{a}}{r}.$$

Obviously (6) may be interpreted as the vector potential of a magnetic dipole of moment $\dfrac{4\pi}{\mu}\mathbf{a}$.

Either (5) or (6) may be applied to the integration of (4) but the necessary transformations turn out to be simpler in the case of (5) in spite of the divergence trouble. We have in fact

$$\text{(7)}\qquad \nabla\times\mathbf{Q} = \nabla\left(\frac{1}{r}\right)\times\mathbf{a}, \qquad \nabla\times\nabla\times\mathbf{Q} = \nabla\left[\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\right],$$

$$\text{(8)}\qquad \mathbf{P}\cdot\nabla\times\nabla\times\mathbf{Q} = \mathbf{A}\cdot\nabla\left[\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\right] = \nabla\cdot\left[\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\mathbf{A}\right].$$

In these transformations and those that follow it is to be kept in mind that **a** is a constant vector. The left-hand side of the identity (3) can now be written

$$\text{(9)}\qquad \int_V \left\{\frac{\mathbf{a}}{r}\cdot\mathbf{J} - \nabla\cdot\left[\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\mathbf{A}\right]\right\}dv = \mathbf{a}\cdot\int_V \frac{\mathbf{J}}{r}\,dv - \mathbf{a}\cdot\int_S (\mathbf{A}\cdot\mathbf{n})\nabla\left(\frac{1}{r}\right)da.$$

Proceeding to the transformation of the surface integrals, we have

$$(10)\quad (\mathbf{P}\times\nabla\times\mathbf{Q})\cdot\mathbf{n} = \left\{\mathbf{A}\times\left[\nabla\left(\frac{1}{r}\right)\times\mathbf{a}\right]\right\}\cdot\mathbf{n} = \mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\times(\mathbf{A}\times\mathbf{n}),$$

$$(11)\quad (\mathbf{Q}\times\nabla\times\mathbf{P})\cdot\mathbf{n} = \left(\frac{\mathbf{a}}{r}\times\nabla\times\mathbf{A}\right)\cdot\mathbf{n} = \mathbf{a}\cdot\frac{\mathbf{B}\times\mathbf{n}}{r},$$

in which $\mathbf{B}$ replaces $\nabla\times\mathbf{A}$. The identity (3) becomes

$$(12)\quad \mu\int_V \frac{\mathbf{J}}{r}\,dv = \int_S (\mathbf{A}\cdot\mathbf{n})\nabla\left(\frac{1}{r}\right)da + \int_S \nabla\left(\frac{1}{r}\right)\times(\mathbf{A}\times\mathbf{n})\,da + \int_S \frac{\mathbf{n}\times\mathbf{B}}{r}\,da.$$

Now the validity of this relation has been established only for regions within which both $\mathbf{P}$ and $\mathbf{Q}$ are continuous and possess continuous first and second derivatives. $\mathbf{Q}$, however, has a singularity at $r = 0$ and consequently this point must be excluded. About the point (x', y', z') a small sphere of radius r_1 is circumscribed. The volume V is now bounded by the surface S_1 of the sphere and an outer enveloping surface S as indicated in Fig. 25, page 166. Since $\nabla(1/r) = \mathbf{r}^0/r^2$, the surface integrals over S_1 may be written

$$\frac{1}{r_1^2}\int_{S_1}\mathbf{r}^0(\mathbf{A}\cdot\mathbf{n})\,da + \frac{1}{r_1^2}\int_{S_1}\mathbf{r}^0\times(\mathbf{A}\times\mathbf{n})\,da + \frac{1}{r_1}\int_{S_1}\mathbf{n}\times\mathbf{B}\,da.$$

The integrand of the middle term is transformed to

$$(13)\quad \mathbf{r}^0\times(\mathbf{A}\times\mathbf{n}) = (\mathbf{r}^0\cdot\mathbf{n})\mathbf{A} - (\mathbf{A}\cdot\mathbf{n})\mathbf{r}^0 + \mathbf{A}\times(\mathbf{r}^0\times\mathbf{n}),$$

and since on the sphere $\mathbf{r}^0\cdot\mathbf{n} = 1$, $\mathbf{r}^0\times\mathbf{n} = 0$, the surface integrals over S_1 reduce to

$$\frac{1}{r_1^2}\int_{S_1}\mathbf{A}\,da + \frac{1}{r_1}\int_{S_1}\mathbf{n}\times\mathbf{B}\,da.$$

If $\bar{\mathbf{A}}$ and $\overline{\mathbf{n}\times\mathbf{B}}$ denote the mean values of the vectors $\mathbf{A}$ and $\mathbf{n}\times\mathbf{B}$ over the surface of the sphere, these integrals have the value

$$\frac{\bar{\mathbf{A}}}{r_1^2}4\pi r_1^2 + \frac{\overline{\mathbf{n}\times\mathbf{B}}}{r_1}4\pi r_1^2,$$

which in the limit as $r_1 \to 0$ reduces to $4\pi\mathbf{A}(x', y', z')$. Upon introducing this result into (12) and transposing, we find the value of the vector potential at any fixed point expressed in terms of a volume integral and of surface integrals over an outer boundary which is again denoted by S.

$$(14)\quad \mathbf{A}(x', y', z') = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(x, y, z)}{r}\,dv - \frac{1}{4\pi}\int_S \frac{\mathbf{n}\times\mathbf{B}}{r}\,da - \frac{1}{4\pi}\int_S (\mathbf{n}\times\mathbf{A})\times\nabla\left(\frac{1}{r}\right)da - \frac{1}{4\pi}\int_S (\mathbf{n}\cdot\mathbf{A})\nabla\left(\frac{1}{r}\right)da.$$

The proof that the divergence of (14) at the point (x', y', z') is zero is left to the reader. Clearly the surface integrals represent the contribution to the vector potential of all sources that are exterior to the surface S. At all points within V, the vector $\mathbf{A}(x', y', z')$ defined by (14) is continuous and has continuous derivatives of all orders. Across the surface S, however, it is apparent from the form of the surface integrals that $\mathbf{A}$ and its derivatives will exhibit certain discontinuities. We shall show in fact that outside S the vector $\mathbf{A}$ is zero everywhere.

The first surface integral in (14) may be interpreted as the contribution to the vector potential of a surface current

$$(15)\quad \mathbf{K} = -\frac{1}{\mu}\mathbf{n}\times\mathbf{B}_-,$$

in which the subscript of $\mathbf{B}_-$ emphasizes that this value of $\mathbf{B}$ is taken just inside the surface S. Now in Sec. 4.12 it was shown that a surface layer of current does not affect the transition of the vector potential, but gives rise to a discontinuity in $\mathbf{B}$ of amount

$$(16)\quad \mathbf{n}\times(\mathbf{B}_+ - \mathbf{B}_-) = \mu\mathbf{K}.$$

Upon replacing $\mathbf{K}$ by its value from (15) it is clear that just outside S

$$(17)\quad \mathbf{n}\times\mathbf{B}_+ = 0$$

In like manner the second surface integral is equivalent to the vector potential of a distribution of magnetic surface polarization of density

$$(18)\quad \mathsf{M} = \frac{1}{\mu}\mathbf{A}_-\times\mathbf{n}.$$

The normal component of $\mathbf{A}$ passes continuously through such a layer, but the tangential component is reduced discontinuously to zero, as we see on substituting (18) into (17), page 247

$$(19)\quad \mathbf{n}\times(\mathbf{A}_+ - \mathbf{A}_-) = -\mathbf{n}\times\mathbf{A}_- + [\mathbf{n}\cdot(\mathbf{n}\times\mathbf{A}_-)]\mathbf{n},$$

whence

$$(20)\quad \mathbf{n}\times\mathbf{A}_+ = 0.$$

The mathematical significance of the last term of (14) is apparent, but it is difficult to imagine a physical distribution of current or magnetic moment of the type called for. This is the form of integral which we have associated with the field intensity of a surface charge of density $(\mathbf{A}_- \cdot \mathbf{n})$, and which leads to a discontinuity in the normal component specified by

$$\mathbf{n} \cdot (\mathbf{A}_+ - \mathbf{A}_-) = \mathbf{n} \cdot \mathbf{A}_-, \tag{21}$$

as a consequence of which we conclude that

$$\mathbf{n} \cdot \mathbf{A}_+ = 0. \tag{22}$$

Thus far we have demonstrated that on the positive side of the closed surface S the tangential and normal components of $\mathbf{A}$ and the tangential component of $\mathbf{B}$ are everywhere zero. It follows at once, however, that the normal component of $\mathbf{B}$ must also vanish over the positive side of S; for the normal component of the curl $\mathbf{A}$ involves only partial derivatives in directions tangential to the surface. Furthermore, we need but apply (14) itself to the region external to S to prove that $\mathbf{A}$, and consequently $\mathbf{B}$, must vanish everywhere. Current and magnetic polarization are absent outside V, since their effect is represented by the surface integrals. Then, since $\mathbf{n} \times \mathbf{B}_+$, $\mathbf{n} \times \mathbf{A}_+$ and $\mathbf{n} \cdot \mathbf{A}_+$ are all zero, it follows from (14) that $\mathbf{A}(x', y', z')$ must vanish at all points outside S.

When $\mathbf{Q} = \nabla(1/r) \times \mathbf{a}$ is chosen in place of (5) as a Green's function, it can be shown without great difficulty that

$$\mathbf{B}(x', y', z') = \frac{\mu}{4\pi} \int_V \mathbf{J} \times \nabla\left(\frac{1}{r}\right) dv - \frac{1}{4\pi} \int_S (\mathbf{n} \times \mathbf{B}) \times \nabla\left(\frac{1}{r}\right) da - \frac{1}{4\pi} \int_S (\mathbf{n} \cdot \mathbf{B}) \nabla\left(\frac{1}{r}\right) da. \tag{23}$$

This is the extension of the Biot-Savart law to a region of finite extent bounded by a surface S. The contribution of currents or magnetic matter outside S to the field within is accounted for by the two surface integrals.

BOUNDARY-VALUE PROBLEMS

4.16. Formulation of the Magnetostatic Problem.—A homogeneous, isotropic body is introduced into the constant field of a fixed and specified system of currents or permanent magnets. Our problem is to determine the resultant field both inside and outside the body. In case the current density at all points within the body is zero, the secondary field arising from the induced magnetization can be represented everywhere by a single-valued scalar potential ϕ_1^* and the methods developed for the treatment of electrostatic problems apply in full. A schedule for the solution may be drawn up as follows.

The scalar potential of the primary field is ϕ_0^*. In case the primary source is a current distribution $\phi_{0\text{I}}^*$ is multivalued but this in no way affects the determination of the induced field ϕ_{II}^*. The resultant potential is $\phi^* = \phi_0^* + \phi_1^*$. The permeability of the body will be denoted by μ_1 and that of the homogeneous medium in which it is embedded by μ_2. Then a function ϕ_1^* must be constructed such that:

(1) $\nabla^2\phi_1^* = 0$, *at all points not on the boundary;*
(2) ϕ_1^* *is finite and continuous everywhere including the boundary;*
(3) *Across the boundary the normal derivatives of the resultant potential* ϕ^* *satisfy the condition*

$$\mu_2\left(\frac{\partial\phi^*}{\partial n}\right)_+ - \mu_1\left(\frac{\partial\phi^*}{\partial n}\right)_- = 0,$$

the subscripts + and − implying that the derivative is calculated outside or inside the boundary surface respectively. The induced potential ϕ_1^* *is, therefore, subject to the condition*

$$\mu_2\left(\frac{\partial\phi_1^*}{\partial n}\right)_+ - \mu_1\left(\frac{\partial\phi_1^*}{\partial n}\right)_- = (\mu_1 - \mu_2)\frac{\partial\phi_0^*}{\partial n} = f,$$

in which f is a known function of position on the boundary satisfying the condition

(4) $\int_S f\, da = 0;$
(5) *At infinity* ϕ_1^* *must vanish at least as* $1/r^2$, *so that* $r^2\phi_1^*$ *remains finite as* $r \to \infty$, *for there is no free magnetic charge and consequently* ϕ_1^* *must vanish as the potential of a dipole or multipole of higher order.*

In case the body carries a current, the interior field cannot be represented by a scalar potential and the boundary-value problem must be solved in terms of a vector potential. Such a case arises, for example, when an iron wire carrying a current is introduced into an external magnetic field. The distribution of the current in the stationary state is unaffected by the magnetic field. Its determination is in fact an electrostatic problem. The vector potential of the primary sources is $\mathbf{A}_0$ while the potential of the induced and permanent magnetization of the body and of the current which it may carry will be denoted by $\mathbf{A}_1$. This function $\mathbf{A}_1$ is subject to the following conditions:

(1) $\nabla \times \nabla \times \mathbf{A}_1 = \mu_1\mathbf{J}$, *at points inside the body where the current density is* $\mathbf{J}$;

(2) $\nabla \times \nabla \times \mathbf{A}_1 = 0$, *at all other points not on the boundary* S;
(3) $\nabla \cdot \mathbf{A}_1 = 0$, *at all points not on* S;
(4) $\mathbf{A}_1$ *is finite and continuous everywhere and passes continuously through the boundary surface* (*Secs.* 4.10 *and* 4.12);
(5) *Across the boundary the normal derivatives of the potential* $\mathbf{A}_1$*—as well as those of the total potential* $\mathbf{A}$*—satisfy*

$$\left(\frac{\partial \mathbf{A}_1}{\partial n}\right)_+ - \left(\frac{\partial \mathbf{A}_1}{\partial n}\right)_- = \mu_0(\mathbf{M}_+ - \mathbf{M}_-) \times \mathbf{n},$$

in which $\mathbf{n}$ *is the outward normal and where* $\mathbf{M}_-$ *is the polarization of the body and* $\mathbf{M}_+$ *that of the medium just outside the boundary.*

Since $\mathbf{M}_+$ and $\mathbf{M}_-$ are determined at least in part by the field itself this relation is usually of no assistance in the determination of $\mathbf{A}_1$. In its place the customary boundary condition on the tangential components of the *total* field must be applied.

(6) $\mathbf{n} \times (\mathbf{H}_+ - \mathbf{H}_-) = \mathbf{n} \times \left(\mathbf{B}_+ - \frac{\mu_2}{\mu_1}\mathbf{B}_-\right) = 0$, *which imposes a relation between the derivatives of* $\mathbf{A}$ *in a specified coordinate system;*
(7) *As* $r \to \infty$ *the product* $r\mathbf{A}_1$ *remains finite.*

4.17. Uniqueness of Solution.—The proof that there is only one function ϕ_1^* satisfying the conditions scheduled above was presented in Sec. 3.20. A corresponding uniqueness theorem for the vector potential may be deduced from the identity (2) of page 250. Let us put $\mathbf{P} = \mathbf{Q} = \mathbf{A}$ and assume first that within V_1 bounded by S the current density is zero. Then $\nabla \times \nabla \times \mathbf{A} = 0$ and

$$\int_{V_1} (\nabla \times \mathbf{A})^2 \, dv = -\int_S \mathbf{A} \cdot (\mathbf{n} \times \nabla \times \mathbf{A}) \, da. \tag{1}$$

From the essentially positive character of the integrand on the left it follows that if $\mathbf{A}$ is zero over the surface S, then $\mathbf{B} = \nabla \times \mathbf{A}$ is zero everywhere within the volume V_1. Hence $\mathbf{A}$ is either constant or at most equal to the gradient of some scalar ψ. But since $\mathbf{A}$ is zero on S, the normal derivative $\partial\psi/\partial n$ is also zero over this surface and it was shown in Sec. 3.20 that this condition entails a constant value of ψ throughout V_1. Consequently, if $\mathbf{A}$ vanishes over a closed surface, it vanishes also at every point of the interior volume. It is also clear that the vector function $\mathbf{A}$ is uniquely determined in V_1 by its values on S. For if there existed two vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ which assumed the specified values over the boundary, their difference must vanish not only over S but also throughout V_1.

The condition $\nabla \times \mathbf{A} = 0$, $\mathbf{A} = \nabla\psi$, throughout V_1 can be established also by the vanishing of $\nabla \times \mathbf{A}$, or of the tangential vector $\mathbf{n} \times \nabla \times \mathbf{A}$ over S. In this case, however, it does not necessarily follow that $\mathbf{A}$ is everywhere zero. If the two functions $\nabla \times \mathbf{A}_1 = \mathbf{B}_1$ and $\nabla \times \mathbf{A}_2 = \mathbf{B}_2$ are identical on S, then $\mathbf{B}_1$ and $\mathbf{B}_2$ are identical at all interior points and $\mathbf{A}_1$ and $\mathbf{A}_2$ can differ at most by the gradient of a scalar function.

In case there are currents present within V_1 the vector potential $\mathbf{A}$ is resolved into a part $\mathbf{A}'$ due to these currents and a part $\mathbf{A}''$ due to external sources. The vector $\mathbf{A}'$ is uniquely determined by the current distribution, while the values of $\mathbf{A}''$ or its curl over S determine $\mathbf{A}''$ at all interior points.

The vector potential is regular at infinity and consequently the proof applies directly to the region V_2 exterior to S. *The vector* $\mathbf{B}$ *is uniquely determined within any domain by the values of its tangential component* $\mathbf{n} \times \mathbf{B}$ *over the boundary.*

PROBLEM OF THE ELLIPSOID

4.18. Field of a Uniformly Magnetized Ellipsoid.—An ellipsoid whose semiprincipal axes are a, b, c is uniformly and permanently magnetized. The direction of magnetization is arbitrary, but since the magnetization vector can be resolved into three components parallel to the principal axes we need consider only the case in which $\mathbf{M}_0$ is constant and parallel to the a-axis.

In view of the uniformity of magnetization $\rho^* = -\nabla \cdot \mathbf{M}_0 = 0$ at all points inside the ellipsoid. The potential ϕ^* of the magnet is due to a "surface charge" of density $\omega^* = \mathbf{n} \cdot \mathbf{M}_0$. The external medium is assumed in this case to be empty space. The problem is now fully equivalent to that of the polarized dielectric ellipsoid treated in Sec. 3.27. From Eqs. (27), (32), and (42) the potential ϕ_1 due to the polarization P_x may be found in terms of P_x and the parameters of the ellipsoid. Upon dropping the factor ϵ_0 and replacing P_x by M_{0x} one obtains

$$\phi_-^* = \frac{abc}{2} A_1 M_{0x} x, \qquad A_1 = \int_0^\infty \frac{ds}{(s + a^2)R_s}, \tag{1}$$

as the magnetic scalar potential at points inside the ellipsoid; at all external points

$$\phi_+^* = \frac{abc}{2} M_{0x} x \int_\xi^\infty \frac{ds}{(s + a^2)R_s}. \tag{2}$$

The field inside the ellipsoid is

$$H_x^- = -\frac{\partial \phi_-^*}{\partial x} = -\frac{abc}{2} A_1 M_{0x}. \tag{3}$$